

# Large-time dynamics and aging of a polymer chain in a random potential

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We study the out-of-equilibrium large-time dynamics of a Gaussian polymer chain in a quenched random potential. The dynamics studied is a simple Langevin dynamics commonly referred to as the Rouse model. The equations for the two-time correlation and response functions are derived within the Gaussian variational approximation. In order to implement this approximation faithfully, we employ the supersymmetric representation of the Martin-Siggia-Rose dynamical action. For a short-ranged correlated random potential the equations are solved analytically in the limit of large times using certain assumptions concerning the asymptotic behavior. Two possible dynamical behaviors are identified depending upon the time separation: a stationary regime and an aging regime. In the stationary regime time translation invariance holds and so does the fluctuation dissipation theorem. The aging regime which occurs for large time separations of the two-time correlation functions is characterized by a history dependence and the breakdown of certain equilibrium relations. The large-time limit of the equations yields equations among the order parameters that are similar to the equations obtained in statics using replicas. In particular the aging solution corresponds to the broken replica solution. But there is a difference in one equation that leads to important consequences for the solution. The stationary regime corresponds to the motion of the polymer inside a local minimum of the random potential, whereas in the aging regime the polymer hops between different minima. As a by-product we also solve exactly the dynamics of a chain in a random potential with quadratic correlations.

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## I. INTRODUCTION

The behavior of polymer chains in random media has attracted much interest in recent years because of its relevance and applications in diverse fields [1,2]. Besides elucidating the properties of the polymers chains themselves which is of much interest in physical chemistry [3,4] and biology [5], this problem is directly related to the statistical mechanics of a quantum particle in a random potential [6], the behavior of flux lines in superconductors in the presence of columnar defects [7,8], and the problem of diffusion in a random catalytic environment [9]. It was found in Refs. [9–13] that a very long Gaussian chain, immersed in a random medium with very-short-range correlations of the disorder, will typically curl up in some small region of low potential energy. The polymer chain is said to be localized, and for long chains the radius of gyration or the end-to-end distance becomes independent of the chain length ( $R^2 \sim L^0$ ).

Both heuristic arguments [11] and a variational solution of the problem using replicas [12] yielded the dependence of the size of the trapped polymer on the variance of the random potential ( $g$ ) and the logarithm of the volume of the medium ( $\mathcal{V}$ ) such that  $R \sim (g \ln \mathcal{V})^{-1/(4-d)}$  for  $1 \leq d < 4$ . The breaking of replica symmetry was crucial to the derivation of the subtle  $\ln \mathcal{V}$  dependence. In a related paper [6], it was found that a quantum particle in a random environment exhibits glassy behavior at low temperatures. The low-temperature limit for a quantum particle translates into the long-chain limit for polymers. It implies that the free energy landscape of a long chain is typically very complicated and possesses many metastable states. In a recent publication [13] we further utilized this mapping to give a physical interpretation of the localization and glassy behavior of a polymer in a random potential by making a connection with the

localization of a quantum particle in a disordered medium of finite volume. The solvable model of a random potential with long-range quadratic correlations was investigated in Ref. [14]. Subsequently, we treated the case of random obstacles as opposed to a random potential [15] and finally included the effect of a self-avoiding interaction [16,17].

Recently methods have been developed to study analytically the large-time, nonequilibrium behavior of glassy systems [18–20]. Directed polymers and manifolds have been investigated at the mean-field level, and the solution exhibits two asymptotic time regimes: a stationary dynamic regime at large but similar times and a slow-aging regime for large and widely separated times. This large-time solution of the dynamical equations has many features in common with the replica-symmetry-breaking (RSB) solution of the corresponding equations of the statics, although replicas are not actually used and the limit  $n \rightarrow 0$  is avoided. But the equations, in particular for the case of one-step RSB, are not all the same in the large-time limit as the equilibrium equations. This leads to a situation wherein the system fails to reach the ultimate equilibrium state starting from an arbitrary initial condition and ends up in a state with higher free energy than the one found at equilibrium. Thus the importance of the dynamical approach is twofold: to make contact with experiments that exhibit slow relaxation and aging effects [21] and also to serve in some cases as an alternative to the replica approach and the  $n \rightarrow 0$  limit, although this has to be taken with a grain of salt as indicated above.

It is our goal here to extend the previous treatment of large-time dynamics for particles and directed manifolds in quenched random environments [18,19] to the case of real polymer chains. The difference between the case of directed polymers and “real” polymers is mainly in the form of the random potential. If  $s$  denotes the bead number of the chain, for “real” chains different beads at the same spatial locations

should be exposed to the same external potential, whereas for directed polymers different segments always feel a different random potential even at the same transverse position. This is easily made clear for flux lines in a superconductors. If we have point disorder different in different  $x$ - $y$  planes when moving along the  $z$  direction, then the problem corresponds to directed polymers in a quenched random potential. On the other hand, if we have randomly positioned columnar defects—i.e., the random potential is independent of  $z$ —then when projected on a single plane we see that the system maps to “real” polymer chains in a quenched random potential. In addition a real polymer might have self-avoiding interactions among different beads, but these will not be considered in this paper.

In this paper we consider the Langevin dynamics of a single Gaussian chain embedded in a quenched random potential. This Langevin dynamics is referred to in the literature as the Rouse model for a polymer and is the simplest dynamics [22]. For polymers in solutions a more realistic dynamics that reproduces more accurately the experimental results is the Zimm model that takes into account the effect hydrodynamic interactions effects. This kind of dynamics will not be considered in this paper and is a project for future research. Our goal here is to consider the simplest model that renders itself to an approximate analytical solution, of the long times dynamics. Various treatments of polymer dynamics in a random potential have been considered before [23–26] using various approximations, renormalization group treatment, and/or numerical solutions of approximate equations of motion. These papers implicitly assume time translation invariance (TTI) of the dynamical correlation function. They try to extract the behavior of the center-of-mass diffusion coefficient at short and large times, and their conclusions are not always in total agreement. Only recently has the possibility of a two-time dependence of the dynamical correlation function—i.e., aging phenomena—been explored numerically by Monte Carlo simulations [27], but the dependence on the waiting time is not reported in detail in this paper.

Our goal is also to make contact with the previous treatment of directed polymers in random potentials and by using similar methods the difference between real and directed polymers will be elucidated. In addition, we derive the equations for a random potential with general correlations, either short or long range. The equations are derived using the so-called Gaussian variational approximation which is a kind of mean-field treatment. This approximation was first introduced by Feynman in his studies of superfluid helium, and in the context of random systems was first introduced in Refs. [28,29]. In order to implement it faithfully so as to preserve correctly all the symmetries of the dynamical equations we use the supersymmetric (SUSY) formulation of the dynamics [30], implement the variational approximation, and then disentangle SUSY to produce coupled integro-differential equations for the correlation and response functions. Here we follow steps similar to those used by Cugliandolo *et al.* [18,19] and Konkoli *et al.* [20] in their treatment of directed manifolds and random heteropolymers respectively. But because the random potential is implemented differently in our case—i.e., different beads at the same spatial position feel the same potential—the resulting equations are different.

In order to solve the equations analytically at large times we make some assumptions about the limiting behavior of the correlation and response functions at large times. These function depend on two different times. When these times are large but their separation is small compare to the individual times, the behavior depends only on the separation and thus TTI holds as well as the fluctuation dissipation theorem (FDT). On the other hand, when the separation of times becomes very large TTI and FDT break down although a generalized form of FDT still holds. The breakdown of TTI and FDT is one of the main characteristics of the glassy phase and is referred to in the literature as “aging.” Although the dynamical equations are valid for random potentials with quite general correlations, in this paper we solve the equations for the case of short-ranged correlations of the random potential. This case includes the case of  $\delta$ -correlated potential—i.e., the potential at different points in space are uncorrelated—but we include the case of a short correlation length and the case that there is power law decay of the correlation with large enough power. In this case the equilibrium solution involved a one-step RSB as found in Ref. [12]. We also consider in the Appendix the case of long-ranged quadratic correlation of the random potential which is exactly soluble without the variational approximation. The case of the solution for other long-ranged correlated potentials will be considered elsewhere. For the statics this case involves continuous RSB [6], and thus the dynamical *Ansatz* should be different.

## II. THE MODEL

The model is defined as follows. The Langevin dynamics is assumed to be governed by the Hamiltonian  $H[x]$ ,

$$\partial x(s,t)/\partial t = -\partial H[x]/\partial x(s,t) + \eta(s,t), \quad (1)$$

where  $x(s,t)$  is a  $d$ -dimensional vector representing the position of chain bead  $s$  at time  $t$ . Beads are numbered continuously from  $s=0$  to  $s=L$ .  $\eta(s,t)$  is Gaussian noise:

$$\langle \eta(s,t) \eta(s',t') \rangle_T = 2\delta(s-s')\delta(t-t')T \quad (2)$$

due to contact with a heat bath at temperature  $T$ . This dynamics is the simplest dynamics for a polymer chain, referred to in the literature as the Rouse model [22]. The Hamiltonian  $H[x]=H_0[x]+H_{rand}[x]$  contains a deterministic part  $H_0[x]$  and a random part  $H_{rand}[x]$ . The  $H_0[x]$  is defined as

$$H_0[x] = \frac{1}{2} \int_0^L ds \{ M [\partial x(s,t)/\partial s]^2 + \mu x(s,t)^2 \}, \quad (3)$$

where  $M=dT/b_K^2$ ,  $d$  is the number of spatial dimensions, and  $b_K$  is the Kuhn bond length of the polymer. This representation is the simplest representation of a polymer as a Gaussian chain in the continuum approximation. The parameter  $\mu$  plays the role of a finite volume since the polymer is confined by the harmonic potential to a finite region of space. Thus  $\mu \rightarrow 0$  is the large-volume limit and  $|\ln \mu| \propto \ln \mathcal{V}$  for a volume  $\mathcal{V}$  [12]. The random part  $H_{rand}$  describes the interaction between each bead and the external random potential:

$$H_{rand}[x] = \int_0^L ds V(x(s,t)). \quad (4)$$

$V(x)$  is a short-range potential, and for simplicity we take it to have a Gaussian form

$$\langle V(x)V(x') \rangle = \frac{g}{(d\pi\sigma^2)^{d/2}} \exp\left(-\frac{(x-x')^2}{d\sigma^2}\right). \quad (5)$$

$d$  is the dimensionality of the system, and  $\sigma$  parametrizes the range of the potential. In particular, for  $\sigma \rightarrow 0$ ,  $\langle V(x)V(x') \rangle \rightarrow g\delta(x-x')$ , and we recover the potential used in [12]. More generally we can take

$$\langle V(x)V(x') \rangle = -dJ\left(\frac{(x-x')^2}{d}\right), \quad (6)$$

for some function  $J(z)$ . For the case represented by Eq. (5),

$$J(z) = -\frac{g}{d(d\pi\sigma^2)^{d/2}} \exp(-z/\sigma^2). \quad (7)$$

This model admits a stationary solution characterized by a Gibbs distribution. The equilibrium partition function for this solution is given by

$$\begin{aligned} \mathcal{Z} = \int Dx \exp\left(-\frac{1}{2T} \int_0^L ds \{M[\partial x(s)/\partial s]^2 \right. \\ \left. + \mu x(s)^2 + V(x(s))\}\right). \end{aligned} \quad (8)$$

### III. MAPPING TO THE FIELD THEORY

Here we will follow closely the notation of Ref. [20]. Using the standard Martin-Siggia-Rose formalism [31], the dynamical average of any observable can be calculated as

$$\langle O[x, \tilde{x}] \rangle_T = \int Dx D\tilde{x} D\xi D\bar{\xi} O(x, \tilde{x}) e^{-S[x, \tilde{x}, \xi, \bar{\xi}]}, \quad (9)$$

with the following dynamical action:

$$\begin{aligned} S[x, \tilde{x}, \xi, \bar{\xi}] = \int dt ds \left[ -T\tilde{x}(s,t)^2 + \tilde{x}(s,t) \left( \frac{\partial}{\partial t} x(s,t) + \frac{\partial H[x]}{\partial x(s,t)} \right) \right] \\ - \int dt ds \bar{\xi}(s,t) \frac{\partial}{\partial t} \xi(s,t) \\ + \int dt ds ds' \bar{\xi}(s,t) \frac{\partial^2 H[x]}{\partial x(s,t) \partial x(s',t)} \xi(s',t), \end{aligned} \quad (10)$$

where  $\tilde{x}$ ,  $\xi$ , and  $\bar{\xi}$  are auxiliary fields which appear in the formalism. To make for a more compact notation we introduce the superfield  $\Phi$ :

$$\Phi(s, t_1, \theta_1, \bar{\theta}_1) = x(s, t_1) + \bar{\xi}(s, t_1)\theta_1 + \bar{\theta}_1\xi(s, t_1) + \bar{\theta}_1\theta_1\tilde{x}(s, t_1), \quad (11)$$

where  $\theta$  and  $\bar{\theta}$  are Grassmann variables (anticommuting  $c$  numbers). For  $X, X' \in \{\theta, \bar{\theta}, \theta', \bar{\theta}'\}$ ,  $\{X, X'\} = 0$  and  $\int dXX = 1$ , the rest of the integrals being zero. In the following, for practical reasons, the more compact notation  $\Phi(s, 1) \equiv \Phi(s, t_1, \theta_1, \bar{\theta}_1)$  will be used. Also, the integral symbol  $\int d\theta_1 d\bar{\theta}_1 dt_1$  will be denoted by  $\int d1$ .

In SUSY notation Eqs. (9) and (10) translate into Eqs. (12) and (13):

$$\langle O[\Phi] \rangle_T = \int D\Phi O[\Phi] e^{-S[\Phi]}, \quad (12)$$

$$S[\Phi] = S_0[\Phi] + S_{rand}[\Phi], \quad (13)$$

where

$$S_0[\Phi] = (1/2) \int ds d1 ds' d2 \Phi(s, 1) K_{12}^{ss'} \Phi(s', 2), \quad (14)$$

$$S_{rand}[\Phi] = \int d1 ds V(\Phi(s, 1)), \quad (15)$$

and

$$K_{12}^{ss'} \equiv \delta_{12} \delta_{ss'} K_1^s, \quad K_1^s = T[\mu/T - (\partial/\partial s)^2] - D_1^{(2)}, \quad (16)$$

$$D_1^{(2)} = 2T \frac{\partial^2}{\partial \theta_1 \partial \bar{\theta}_1} + 2\theta_1 \frac{\partial^2}{\partial \theta_1 \partial t_1} - \frac{\partial}{\partial t_1}. \quad (17)$$

As noticed by De Dominicis [32] the expression in Eq. (12) is already normalized, so the average over the quenched random interactions  $V$  can be done directly on (12):

$$\langle \langle A[\Phi] \rangle_T \rangle_V = \int D\Phi A[\Phi] e^{-(S_0[\Phi] + S_1[\Phi])}, \quad (18)$$

where  $\exp(-S_1[\Phi]) \equiv \langle \exp(-S_{rand}[\Phi]) \rangle_V$ . The average over  $V$  can be done easily, leading to

$$S_1[\Phi] = \frac{d}{2} \int ds ds' d1 d2 J \left( \frac{[\Phi(s, 1) - \Phi(s', 2)]^2}{d} \right). \quad (19)$$

The dynamical action  $S_{AV} = S_0 + S_1$  closely resembles the effective Hamiltonian obtained in the static replica approach of Refs. [6,12]. This rather general similarity between replica and SUSY treatments has been discussed in Ref. [30]. Instead of summation over replica indices in [6,12] we have  $\int d1 \int d2$ .

### IV. VARIATIONAL APPROXIMATION

Since the model cannot be solved exactly, we proceed by using a variational approximation, first introduced by Feynman. We assume that fields  $\Phi$  are approximately described by a quadratic action

$$S_{var} = \frac{1}{2} \int d1 ds d2 ds' \Phi(s, 1) G(s, 1; s', 2)^{-1} \Phi(s', 2). \quad (20)$$

This approach has been widely used in statics. Here we apply it to a dynamic calculation. The goal is to calculate  $F_{dyn}$ , given formally by

$$e^{-F_{dyn}} = e^{-\langle (S_{AV} - S_{var}) \rangle_{var}} e^{-f_{var}}, \quad (21)$$

where

$$e^{-f_{var}} = \int D\Phi e^{-S_{var}}, \quad \langle \cdot \rangle_{var} = e^{f_{var}} \int D\Phi(\cdot) e^{-S_{var}}. \quad (22)$$

In usual statics, for problems without disorder, the variational approach is related to a maximum principle. The equivalent of Eq. (21) leads to the inequality

$$e^{-F} \geq e^{-\langle (S_{AV} - S_{var}) \rangle_{var}} e^{-f_{var}}. \quad (23)$$

In the present dynamical problem, as well as in the static problem with replicas, such a maximum principle is not known and the variational free energy cannot be claimed to be an upper bound to the true one. Despite that, the variational approach has been argued to give exact results for directed manifolds in the limit of infinite embedding dimensions [29,33], giving some justification for its use even at finite dimensions. For real polymers we obtain a meaningful solution for  $1 \leq d < 4$  and we cannot use the large- $d$  limit directly.

The dynamical variational free energy  $F_{dyn} = \langle (S_{AV} - S_{var}) \rangle_{var} + f_{var}$  is given by

$$F_{dyn} = F_{dyn}^{(1)} + F_{dyn}^{(2)} + F_{dyn}^{(3)}, \quad (24)$$

with

$$F_{dyn}^{(1)} = \frac{d}{2} \int ds d1 ds' d2 K_{12}^{ss'} G_{12}^{ss'}, \quad (25)$$

$$F_{dyn}^{(2)} = -\frac{d}{2} \text{Tr} \ln G, \quad (26)$$

$$F_{dyn}^{(3)} = \frac{d}{2} \int ds d1 ds' d2 \left\langle J \left( \frac{(\Phi(s, 1) - \Phi(s', 2))^2}{d} \right) \right\rangle_{var}. \quad (27)$$

We proceed to calculate the last term  $F_{dyn}^{(3)}$ . Using the identity

$$\begin{aligned} \langle J((\Phi - \Phi')^2/d) \rangle_{var} &= \int d^d y J(y^2/d) \int \frac{d^d p}{(2\pi)^d} \exp(-ip \cdot y) \\ &\times \langle \exp[ip \cdot (\Phi - \Phi')] \rangle_{var}, \end{aligned} \quad (28)$$

it is easy to verify that

$$\langle \exp[ip \cdot (\Phi - \Phi')] \rangle_{var} = \exp\left(-\frac{1}{2} p^2 B_{12}^{ss'}\right), \quad (29)$$

where

$$B_{12}^{ss'} = G(s, 1; s, 1) + G(s', 2; s', 2) - 2G(s, 1; s', 2). \quad (30)$$

Defining

$$\begin{aligned} \hat{J}(a) &\equiv \int d^d y J(y^2/d) \int \frac{d^d p}{(2\pi)^d} \exp(-ip \cdot y) \exp\left(-\frac{ap^2}{2}\right) \\ &= \frac{1}{\Gamma(d/2)} \int_0^\infty dx x^{d/2-1} e^{-x} J\left(\frac{2xa}{d}\right), \end{aligned} \quad (31)$$

we observe, by substituting Eq. (29) in Eq. (28), that

$$\langle J((\Phi - \Phi')^2/d) \rangle_{var} = \hat{J}(B_{12}^{ss'}). \quad (32)$$

Thus

$$F_{dyn}^{(3)} = \frac{d}{2} \int ds d1 ds' d2 \hat{J}(B_{12}^{ss'}). \quad (33)$$

For the case that  $J$  is given by Eq. (7) we find

$$\hat{J}(a) = -\frac{g}{d(2\pi)^{d/2}} \left(\frac{d\sigma^2}{2} + a\right)^{-d/2} \quad (34)$$

and recall that  $\sigma \rightarrow 0$  for a  $\delta$ -correlated potential. If  $J(a)$  is of the form

$$J(a) = \frac{ga^{1-\gamma}}{2(1-\gamma)}, \quad (35)$$

for large  $a$ —i.e., it involves power law correlations of the disorder—then

$$\hat{J}(a) = \frac{\hat{g}a^{1-\hat{\gamma}}}{2(1-\hat{\gamma})}, \quad (36)$$

for large  $a$ , where [29]

$$\hat{\gamma} = \gamma \quad \text{if } \gamma < 1 + d/2, \quad (37)$$

$$\hat{\gamma} = 1 + d/2 \quad \text{if } \gamma \geq 1 + d/2, \quad (38)$$

$$\hat{g} \propto g. \quad (39)$$

## V. EQUATIONS OF MOTION IN SUPERSYMMETRIC NOTATION

Given the  $F_{dyn}$ , one can derive the equations of motion from the stationarity condition

$$\frac{\delta}{\delta G_{12}^{ss'}} F_{dyn} = 0. \quad (40)$$

The most complicated term is  $\frac{\delta}{\delta G_{12}^{ss'}} F_{dyn}^{(3)}$ . From Eq. (33), it is

$$\begin{aligned} \frac{d}{2} \int d3 d4 dudv \hat{J}'(B_{34}^{uv}) &(\delta_{ss'} \delta_{us} \delta_{13} \delta_{23} + \delta_{ss'} \delta_{vs} \delta_{14} \delta_{24} \\ &- \delta_{us} \delta_{vs'} \delta_{13} \delta_{24} - \delta_{us'} \delta_{vs} \delta_{14} \delta_{23}). \end{aligned} \quad (41)$$

Equation (41) simplifies to

$$\frac{\delta}{\delta G_{12}^{ss'}} F_{dyn}^{(3)} = d \left[ \delta_{ss'} \delta_{12} \int d3du \hat{J}'(B_{13}^{su}) - \hat{J}'(B_{12}^{ss'}) \right]. \quad (42)$$

The variations of  $F_{dyn}^{(1)}$  and  $F_{dyn}^{(2)}$  are trivial. Using Eqs. (40) and (24) leads to

$$K_{12}^{ss'} - (G_{12}^{ss'})^{-1} + 2 \left[ \delta_{ss'} \delta_{12} \int d3du \hat{J}'(B_{13}^{su}) - \hat{J}'(B_{12}^{ss'}) \right] = 0, \quad (43)$$

which can be written as

$$K_1^s G_{12}^{ss'} = \delta_{12} \delta_{ss'} + 2 \int d3du \hat{J}'(B_{13}^{su}) (G_{32}^{us'} - G_{12}^{ss'}). \quad (44)$$

Due to translational invariance in the variable  $s$ ,  $G$  depends only on the difference  $s-s'$ . Thus

$$G_{12}^{ss'} = G_{12}^{s-s'}. \quad (45)$$

It is useful to define following Fourier transforms:

$$G_{12}^s \equiv \frac{1}{L} \sum_k e^{-iks} \hat{G}_{12}^k. \quad (46)$$

Since  $0 < s < L$ , the corresponding wave numbers  $k$  take the values  $k = (2\pi/L)n$  where  $n = 0, \pm 1, \pm 2, \dots$ . In the following it will become necessary to separate the  $k=0$  component from  $k \neq 0$ . Also

$$\hat{G}_{12}^k = \int_0^L ds e^{iks} G_{12}^s. \quad (47)$$

Then Eq. (44) translates into

$$[\mu + Tk^2 - D_1^{(2)}] \hat{G}_{12}^k = \delta_{12} + 2 \int d3du \hat{J}'(B_{13}^{su}) (e^{iku} \hat{G}_{32}^k - \hat{G}_{12}^k), \quad (48)$$

where

$$B_{13}^u = \frac{1}{L} \sum_{k'} (\hat{G}_{11}^{k'} + \hat{G}_{33}^{k'} - 2e^{-ik'u} \hat{G}_{13}^{k'}). \quad (49)$$

## VI. DISENTANGLING SUPERSYMMETRY

$G_{12}^{ss'}$  encodes 16 correlation functions, out of which only two, the correlation and response function, are independent and nonzero:

$$\langle \langle x(s, t_1) x(s', t_2) \rangle \rangle / d \equiv C(s, t_1; s', t_2) = \frac{1}{L} \sum_k e^{ik(s-s')} C_k(t_1, t_2), \quad (50)$$

$$\langle \langle x(s, t_1) \tilde{x}(s', t_2) \rangle \rangle / d \equiv R(s, t_1; s', t_2) = \frac{1}{L} \sum_k e^{ik(s-s')} R_k(t_1, t_2). \quad (51)$$

Also, by adding an external field term to the original Hamiltonian  $H[x] \rightarrow H[x] + \int ds dt x(s, t) h(s, t)$  one gets

$$\langle \langle x(s, t_1) \tilde{x}(s', t_2) \rangle \rangle = \frac{\delta}{\delta h(s', t_2)} \langle \langle x(s, t_1) \rangle \rangle; \quad (52)$$

i.e.,  $R(s, t_1; s', t_2)$  describes the response to an infinitesimal field applied at time  $t_2$  and bead  $s'$ . Thus,  $G_{12}^k$  reduces to

$$G_{12}^k = C_k(t_1, t_2) + (\bar{\theta}_1 - \bar{\theta}_2) [\theta_1 R_k(t_2, t_1) - \theta_2 R_k(t_1, t_2)]. \quad (53)$$

It follows that

$$B_{12}^u = B^u(t_1, t_2) - \frac{2}{L} \sum_k e^{-ik'u} (\bar{\theta}_1 - \bar{\theta}_2) [\theta_1 R_{k'}(t_2, t_1) - \theta_2 R_{k'}(t_1, t_2)], \quad (54)$$

with

$$B^u(t_1, t_2) \equiv \langle \langle [x(u, t_1) - x(0, t_2)]^2 \rangle \rangle / d = \frac{1}{L} \sum_k [C_k(t_1, t_1) + C_k(t_2, t_2) - 2e^{-iku} C_k(t_1, t_2)]. \quad (55)$$

Disentangling the equations of motion in SUSY notation [see Eq. (48)] by using Eqs. (53)–(55) gives

$$\begin{aligned} & [\mu + Mk^2 + \partial/\partial t] C_k(t, t') \\ &= 2TR_k(t', t) + 2 \int_0^{t'} dt_3 \int_0^L du \hat{J}'(B^u(t, t_3)) R_k(t', t_3) e^{iku} \\ &+ 4 \int_0^t dt_3 \int_0^L du \hat{J}''(B^u(t, t_3)) R^u(t, t_3) \\ &\times [C_k(t, t') - e^{iku} C_k(t_3, t')], \end{aligned} \quad (56)$$

$$\begin{aligned} & [\mu + Mk^2 + \partial/\partial t] R_k(t, t') \\ &= \delta(t - t') + 4 \int_0^t dt_3 \int_0^L du \hat{J}''(B^u(t, t_3)) \\ &\times R^u(t, t_3) [R_k(t, t') - e^{iku} R_k(t_3, t')], \end{aligned} \quad (57)$$

where we defined

$$R^u(t, t') = \frac{1}{L} \sum_k e^{-iku} R_k(t, t'). \quad (58)$$

## VII. ANSATZ FOR THE CORRELATION AND RESPONSE FUNCTIONS

These equations of motion are coupled integro-differential equations which in principle can be solved; the initial conditions are given by  $C_k(0, 0)$  and we use Ito's convention  $R(t + \epsilon, t) \rightarrow 1$  as  $\epsilon \rightarrow 0$  from above. It is well known that asymptotic solutions of such equations can be characterized by few parameters and it is possible to solve those equations analytically [18, 19, 34–37].



For  $t, t' \rightarrow \infty$ ,  $\tau/t' \ll 1$  and  $\tau = t - t'$ , TTI holds:

$$\lim_{t \rightarrow \infty} C_k(t, t) = \tilde{q}_k, \quad (59)$$

$$\lim_{t \rightarrow \infty} C_k(t + \tau, t) = C_k(\tau), \quad (60)$$

$$\lim_{\tau \rightarrow \infty} C_k(\tau) = q_k, \quad (61)$$

and

$$\lim_{t \rightarrow \infty} R_k(t + \tau, t) = R_k(\tau). \quad (62)$$

We will refer to this regime as the stationary or TTI regime. In addition to the TTI regime, there is another long-time nontrivial regime, characterized by  $t, t' \rightarrow \infty$ , fixing  $\alpha = h(t')/h(t)$  and  $0 < \alpha < 1$ , where the function  $h(t)$  is an increasing function of  $t$  which the asymptotic analysis performed here is not able to determine. In this aging regime one has

$$\lim_{t \rightarrow \infty} C_k(t, h^{-1}(\alpha h(t))) = q_k \hat{C}_k(\alpha), \quad (63)$$

$$\lim_{\alpha \rightarrow 0} q_k \hat{C}_k(\alpha) = q_{0,k}, \quad (64)$$

$$\lim_{\alpha \rightarrow 1} \hat{C}_k(\alpha) = 1, \quad (65)$$

and

$$\lim_{t \rightarrow \infty} R_k(t, \alpha t) = \frac{1}{t} \hat{R}_k(\alpha). \quad (66)$$

Also, for future convenience, it is useful to introduce the following order parameters:

$$\tilde{b}(u) = \frac{2}{L} \sum_k (1 - e^{-iku}) \tilde{q}_k, \quad (67)$$

$$b(u) = \frac{2}{L} \sum_k (\tilde{q}_k - e^{-iku} q_k), \quad (68)$$

$$b_0(u) = \frac{2}{L} \sum_k (\tilde{q}_k - e^{-iku} q_{0,k}). \quad (69)$$

Also

$$B^u(\tau) = \frac{2}{L} \sum_k [\tilde{q}_k - e^{-iku} C_k(\tau)], \quad (70)$$

$$\hat{B}^u(\alpha) = \frac{2}{L} \sum_k [\tilde{q}_k - e^{-iku} q_k \hat{C}_k(\alpha)], \quad (71)$$

$$\hat{R}^u(\alpha) = \frac{1}{L} \sum_k e^{-iku} \hat{R}_k(\alpha). \quad (72)$$

## VIII. EQUATIONS RELATING ASYMPTOTIC VALUES OF CORRELATION AND RESPONSE FUNCTIONS

Using the *Ansatz* discussed in Sec. VII one can derive the following equations for  $C_k(t, t')$  in the TTI regime:

$$\begin{aligned} & (\mu + Tk^2 + \partial/\partial\tau) C_k(\tau) \\ &= 2TR_k(-\tau) + \frac{2}{T} \int_0^L du \hat{J}'(b(u)) [C_k(\tau) - e^{iku} q_k] \\ & - \frac{2}{T} \int_0^L du \hat{J}'(\tilde{b}(u)) (1 - e^{iku}) C_k(\tau) \\ & - \frac{2}{T} \int_0^\tau d\tau' \int_0^L du e^{iku} \hat{J}'(B^u(\tau - \tau')) \frac{\partial C_k(\tau')}{\partial \tau'} \\ & + 2 \int_0^1 d\rho \int_0^L du e^{iku} \hat{J}'(\hat{B}^u(\rho)) \hat{R}_k(\rho) \\ & + 4 \int_0^1 d\rho \int_0^L du \hat{J}''(\hat{B}^u(\rho)) \hat{R}^u(\rho) \\ & \times [C_k(\tau) - e^{iku} q_k \hat{C}_k(\rho)]. \end{aligned} \quad (73)$$

It is also possible to derive similar equations for  $R_k(\tau)$  which, due to the FDT

$$R_k(\tau) = -\frac{1}{T} \frac{dC_k(\tau)}{d\tau}, \quad (74)$$

are completely equivalent to Eq. (73).

In the aging regime one gets the following equation for  $q_k \hat{C}_k(\alpha)$ :

$$\begin{aligned} & \left[ \mu + Mk^2 - 4 \int_0^1 d\rho \int_0^L du e^{iku} \hat{J}''(\hat{B}^u(\rho)) \hat{R}^u(\rho) \right] q_k \hat{C}_k(\alpha) \\ &= 2 \int_0^1 d\rho \int_0^L du e^{iku} \hat{J}'(\hat{B}^u(\alpha\rho)) \hat{R}_k(\rho) \\ & + \frac{2}{T} \int_0^L du e^{iku} \hat{J}'(\hat{B}^u(\alpha)) (\tilde{q}_k - q_k) \\ & - 4 \int_0^\alpha d\rho \int_0^L du e^{iku} \hat{J}''(\hat{B}^u(\rho)) \hat{R}^u(\rho) q_k \hat{C}_k(\rho/\alpha) \\ & - 4 \int_\alpha^1 d\rho \int_0^L du e^{iku} \hat{J}''(\hat{B}^u(\rho)) \hat{R}^u(\rho) q_k \hat{C}_k(\alpha/\rho) \\ & + 4 \int_0^\infty d\tau' \int_0^L du \hat{J}''(B^u(\tau')) R^u(\tau') (1 - e^{iku}) q_k \hat{C}_k(\alpha). \end{aligned} \quad (75)$$

For  $\hat{R}_k(\alpha)$  we obtain

$$\begin{aligned}
& \left[ \mu + Mk^2 - 4 \int_0^1 d\rho \int_0^L du \hat{J}''(\hat{B}^u(\rho)) \hat{R}^u(\rho) \right] \hat{R}_k(\alpha) \\
&= -\frac{4}{T} \int_0^L du e^{iku} \hat{J}''(\hat{B}(\alpha)) \hat{R}^u(\alpha) (\tilde{q}_k - q_k) \\
&\quad - 4 \int_\alpha^1 \frac{d\rho}{\rho} \int_0^L du e^{iku} \hat{J}''(\hat{B}^u(\rho)) \hat{R}^u(\rho) \hat{R}_k(\alpha/\rho) \\
&\quad + 4 \int_0^\infty d\tau' \int_0^L du (1 - e^{iku}) \hat{J}''(\hat{B}^u(\tau')) \hat{R}^u(\tau') \hat{R}_k(\alpha).
\end{aligned} \tag{76}$$

Again, one can see that both Eqs. (75) and (76) can be solved by the *Ansatz*

$$\hat{R}_k(\alpha) = \frac{x_c}{T} q_k \frac{d\hat{C}_k(\alpha)}{d\alpha}. \tag{77}$$

Equation (77) is commonly referred to as a generalized FDT (GFDT). The parameter  $x_c$  corresponds to the corresponding parameter in the static replica solution. In the context of replicas it was first introduced by Parisi and should not be confused with a spatial coordinate. This parameter must satisfy the inequality  $x_c < 1$ . In principle, Eq. (77) could have been written as

$$\hat{R}_k(\alpha) = \frac{x_{ck}(q_k \hat{C}_k(\alpha))}{T} q_k \frac{d\hat{C}_k(\alpha)}{d\alpha}, \tag{78}$$

which could be applied to a many-step RSB scheme. However, as for the case of directed polymer with short ranged correlated random potential we found [12] that a solution with one step RSB is appropriate, and it is sufficient to use the simpler *Ansatz* given in Eq. (77).

For  $t=t'$  and  $t \rightarrow \infty$ , Eq. (56) gives

$$\begin{aligned}
(\mu + Mk^2) \tilde{q}_k &= T + \frac{2}{T} \int_0^L du \hat{J}'(b(u)) (\tilde{q}_k - e^{iku} q_k) \\
&\quad - \frac{2}{T} \int_0^L du (1 - e^{iku}) \hat{J}'(\tilde{b}(u)) \tilde{q}_k \\
&\quad + 2 \int_0^1 d\rho \int_0^L du e^{iku} \hat{J}'(\hat{B}^u(\rho)) \hat{R}_k(\rho) \\
&\quad + 4 \int_0^1 d\rho \int_0^L du \hat{J}''(\hat{B}^u(\rho)) \hat{R}^u(\rho) \\
&\quad \times [\tilde{q}_k - e^{iku} q_k C_k(\rho)].
\end{aligned} \tag{79}$$

Equation (73) for  $t \rightarrow \infty$  and then  $\tau \rightarrow \infty$  results in

$$\begin{aligned}
(\mu + Mk^2) q_k &= \frac{2}{T} \int_0^L du (1 - e^{iku}) \{ \hat{J}'(b(u)) - \hat{J}'(\tilde{b}(u)) \} q_k \\
&\quad + \frac{2}{T} \int_0^L du e^{iku} \hat{J}'(b(u)) (\tilde{q}_k - q_k) \\
&\quad + 2 \int_0^1 d\rho \int_0^L du e^{iku} \hat{J}'(\hat{B}^u(\rho)) \hat{R}_k(\rho) \\
&\quad + 4 \int_0^1 d\rho \int_0^L du \hat{J}''(\hat{B}^u(\rho)) \hat{R}^u(\rho) q_k \\
&\quad \times [1 - e^{iku} C_k(\rho)].
\end{aligned} \tag{80}$$

Also, Eq. (75) for  $\alpha \rightarrow 0$  gives

$$\begin{aligned}
(\mu + Mk^2) q_{0,k} &= 2 \int_0^L du e^{iku} \hat{J}'(b_0(u)) \int_0^1 d\rho \hat{R}_k(\rho) \\
&\quad + \frac{2}{T} \int_0^L du e^{iku} \hat{J}'(b_0(u)) (\tilde{q}_k - q_k) \\
&\quad + \frac{2}{T} \int_0^L du (1 - e^{iku}) \{ \hat{J}'(b(u)) - \hat{J}'(\tilde{b}(u)) \} \\
&\quad \times q_{0,k} + \frac{4}{T} \int_0^L du (1 - e^{iku}) \hat{J}''(\hat{B}^u(\rho)) \hat{R}^u(\rho) q_{0,k}.
\end{aligned} \tag{81}$$

Equations (79)–(81) and their origin, Eqs. (73), (75), and (76), contain both TTI and aging parts. Thus, in principle, there are two *Ansätze* for solving them, leading to two physical behaviors: an ergodic phase (this name was coined by KHF [20] in the sense of nonglassy) characterized by TTI and FDT where the aging behavior is completely missing, and a glassy phase containing both stationary and aging behaviors.

## IX. ERGODIC PHASE

By an ergodic phase we mean that the external parameters are such that only the stationary solution exists and not the aging solution. This happens when the only solution is with  $q_k = q_{0,k}$ . Technically, assuming that aging is absent amounts to setting  $\hat{R}_k(\alpha) = 0$  and  $\hat{C}_k(\alpha) = 1$  in Eqs. (79)–(81). [Equivalently, one could start from Eqs. (56) and (57) and exclude the aging part from the beginning, leading to the same equations.] Thus, in the ergodic phase, Eqs. (79)–(81) reduce to

$$\begin{aligned}
(\mu + Mk^2) \tilde{q}_k &= T + \frac{2}{T} \int_0^L du \hat{J}'(b(u)) (\tilde{q}_k - e^{iku} q_k) \\
&\quad - \frac{2}{T} \int_0^L du (1 - e^{iku}) \hat{J}'(\tilde{b}(u)) \tilde{q}_k,
\end{aligned} \tag{82}$$

$$\begin{aligned}
(\mu + Mk^2)q_k &= \frac{2}{T} \int_0^L du e^{iku} \hat{J}'(b(u)) (\tilde{q}_k - q_k) \\
&+ \frac{2}{T} \int_0^L du (1 - e^{iku}) \{ \hat{J}'(b(u)) - \hat{J}'(\tilde{b}(u)) \} q_k,
\end{aligned} \tag{83}$$

$$\begin{aligned}
(\mu + Mk^2)q_{0,k} &= \frac{2}{T} \int_0^L du e^{iku} \hat{J}'(b_0(u)) (\tilde{q}_k - q_k) \\
&+ \frac{2}{T} \int_0^L du (1 - e^{iku}) \{ \hat{J}'(b(u)) - \hat{J}'(\tilde{b}(u)) \} q_{0,k}.
\end{aligned} \tag{84}$$

Note that Eqs. (83) and (84) enforce  $q_k = q_{0,k}$  which is just equivalent to  $\hat{C}_k(\alpha) = 1$ , so one gets only two equations.

In order to solve these equation we define the following constants, which are themselves functions of  $\tilde{q}_k$  and  $q_k$ :

$$E_k = \frac{2}{T} \int_0^L du e^{iku} \hat{J}'(b(u)), \tag{85}$$

$$F_k = \frac{2}{T} \int_0^L du (1 - e^{iku}) \hat{J}'(\tilde{b}(u)). \tag{86}$$

In terms of these constants the equations for  $\tilde{q}_k$  and  $q_k$  become:

$$(\mu + Mk^2)\tilde{q}_k = T + E_0\tilde{q}_k - E_k q_k - F_k \tilde{q}_k, \tag{87}$$

$$(\mu + Mk^2)q_k = E_k(\tilde{q}_k - q_k) + (E_0 - E_k - F_k)q_k. \tag{88}$$

The solution of these equations is

$$\tilde{q}_k = \frac{T(\mu + Mk^2 - E_0 + 2E_k + F_k)}{(\mu + Mk^2 - E_0 + E_k + F_k)^2}, \tag{89}$$

$$q_k = \frac{TE_k}{(\mu + Mk^2 - E_0 + E_k + F_k)^2}. \tag{90}$$

We now show that the *Ansatz*  $E_k = 0$  for  $k \neq 0$  solves the equations. Since  $F_0 = 0$ , we find

$$\tilde{q}_0 = \frac{T}{\mu} + \frac{TE_0}{\mu^2}, \tag{91}$$

$$\tilde{q}_{k \neq 0} = \frac{T}{\mu + Mk^2 - E_0 + F_k}, \tag{92}$$

$$q_0 = \frac{TE_0}{\mu^2}, \quad q_{k \neq 0} = 0. \tag{93}$$

Using these solutions we see that

$$b(u) = \frac{2T}{L\mu} + \frac{2T}{L} \sum_{k \neq 0} \frac{1}{\mu + Mk^2 - E_0 + F_k} \equiv b \tag{94}$$

is independent of  $u$ . Thus

$$E_k = \frac{2}{T} \hat{J}'(b) \int_0^L du e^{iku} = \frac{2L}{T} \hat{J}'(b) \delta_{k,0}, \tag{95}$$

which validates our *Ansatz*. Thus we can write

$$E_0 = \frac{2L}{T} \hat{J}' \left( \frac{2T}{L\mu} + \frac{2T}{L} \sum_{k \neq 0} \frac{1}{\mu + Mk^2 - E_0 + F_k} \right). \tag{96}$$

Also

$$\tilde{b}(u) = \frac{2}{L} \sum_k (1 - e^{-iku}) \tilde{q}_k = \frac{2T}{L} \sum_k \frac{1 - e^{-iku}}{\mu + Mk^2 - E_0 + F_k}, \tag{97}$$

and hence

$$F_k = \frac{2}{T} \int_0^L du (1 - e^{iku}) \hat{J}' \left( \frac{2T}{L} \sum_{k' \neq 0} \frac{1 - e^{-ik'u}}{\mu + Mk'^2 - E_0 + F_{k'}} \right). \tag{98}$$

This equation together with Eq. (96) gives a complete set of equations in the ergodic case. They are identical to Eqs. (4.15) and (4.16) in Ref. [6] derived by the replica method for a quantum particle in a random potential (the notation there is slightly different but it is easy to identify the corresponding variables). Thus in the ergodic case there is a complete agreement between the dynamical calculation and the replica calculation.

From the results above we obtain

$$\begin{aligned}
\tilde{q} &= \lim_{t \rightarrow \infty} \langle \langle x(s,t)x(s,t) \rangle \rangle / d = \frac{1}{L} \sum_k \tilde{q}_k \\
&= \frac{T}{L\mu} + \frac{TE_0}{L\mu^2} + \frac{T}{L} \sum_{k \neq 0} \frac{1}{\mu + Mk^2 - E_0 + F_k}, \\
q &= \lim_{\tau \rightarrow \infty} \lim_{t \rightarrow \infty} \langle \langle x(s,t+\tau)x(s,t) \rangle \rangle / d = \frac{1}{L} \sum_k q_k = \frac{TE_0}{L\mu^2}.
\end{aligned} \tag{99}$$

## X. SPIN-GLASS PHASE

In this phase  $q_0 \neq q_{0,k}$  and there is a time regime where the aging behavior takes place. This phase corresponds to the RSB phase of the statics. We introduce the functions

$$E_{0,k} = \frac{2}{T} \int_0^L du e^{iku} \hat{J}'(b_0(u)). \tag{100}$$

Keeping the aging parts and using the GFDT, Eqs. (79)–(81) can be transformed into

$$\begin{aligned}
(\mu + Mk^2)\tilde{q}_k &= T + E_0(1 - x_c)\tilde{q}_k - E_k(1 - x_c)q_k - F_k \tilde{q}_k \\
&+ E_{0,0}x_c \tilde{q}_k - E_{0,k}x_c q_{0,k},
\end{aligned} \tag{101}$$

$$\begin{aligned}
(\mu + Mk^2)q_k &= E_k(\tilde{q}_k - q_k) + [E_0(1 - x_c) - E_k(1 - x_c) \\
&+ E_{0,0}x_c - F_k]q_k - E_{0,k}x_c q_{0,k},
\end{aligned} \tag{102}$$



$$(\mu + Mk^2)q_{0,k} = E_{0,k}\tilde{q}_k - E_{0,k}(1-x_c)q_k + [E_0(1-x_c) - E_k(1-x_c) + E_{0,0}x_c - 2E_{0,k}x_c - F_k]q_{0,k}. \quad (103)$$

In this case, similar to the ergodic case we will use the *Ansatz*  $E_k=0$  and  $E_{0,k}=0$  for  $k \neq 0$ . We will see that this provides again a consistent solution. Using this *Ansatz* the solution of Eqs. (101)–(103) for  $\tilde{q}_k$ ,  $q_k$ , and  $q_{0,k}$  becomes

$$\tilde{q}_0 = \frac{T}{\mu + \Sigma} + q_0, \quad \tilde{q}_{k \neq 0} = \frac{T}{\mu + Mk^2 - E_0 + \Sigma + F_k}, \quad (104)$$

$$q_0 = \frac{T(\mu E_0 + E_{0,0}\Sigma)}{\mu^2(\mu + \Sigma)}, \quad q_{k \neq 0} = 0, \quad (105)$$

$$q_{0,0} = \frac{TE_{0,0}}{\mu^2}, \quad q_{0,k \neq 0} = 0, \quad (106)$$

where

$$\Sigma = x_c(E_0 - E_{0,0}). \quad (107)$$

Using this solution we see that

$$b(u) = \frac{2T}{L(\mu + \Sigma)} + \frac{2T}{L} \sum_{k \neq 0} \frac{1}{\mu + Mk^2 - E_0 + \Sigma + F_k} \equiv b, \quad (108)$$

$$b_0(u) = \frac{2T(\mu + E_0 - E_{0,0})}{L\mu(\mu + \Sigma)} + \frac{2T}{L} \sum_{k \neq 0} \frac{1}{\mu + Mk^2 - E_0 + \Sigma + F_k} \equiv b_0, \quad (109)$$

both independent of  $u$ . Thus

$$E_k = \frac{2}{T} \hat{J}'(b) \int_0^L du e^{iku} = \frac{2L}{T} \hat{J}'(b) \delta_{k,0}, \quad (110)$$

$$E_{0,k} = \frac{2}{T} \hat{J}'(b_0) \int_0^L du e^{iku} = \frac{2L}{T} \hat{J}'(b_0) \delta_{k,0}, \quad (111)$$

consistent with our *Ansatz*. If we denote

$$a(u) = \frac{2T}{L} \sum_{k \neq 0} \frac{e^{-iku}}{\mu + Mk^2 - E_0 + \Sigma + F_k}, \quad (112)$$

we can write

$$E_0 = \frac{2L}{T} \hat{J}'\left(a(0) + \frac{2T}{L(\mu + \Sigma)}\right), \quad (113)$$

$$E_{0,0} = \frac{2L}{T} \hat{J}'\left(a(0) + \frac{2T}{L(\mu + \Sigma)}\left(1 + \frac{\Sigma}{\mu x_c}\right)\right), \quad (114)$$

and thus

$$\Sigma = \frac{2Lx_c}{T} \left\{ \hat{J}'\left(a(0) + \frac{2T}{L(\mu + \Sigma)}\right) - \hat{J}'\left(a(0) + \frac{2T}{L(\mu + \Sigma)}\left(1 + \frac{\Sigma}{\mu x_c}\right)\right) \right\}. \quad (115)$$

For  $F_k$  we get, instead of Eq. (98),

$$F_k = \frac{2}{T} \int_0^L du (1 - e^{iku}) \hat{J}'(a(0) - a(u)). \quad (116)$$

Let us define the functions

$$D_k = \frac{2}{T} \int_0^L du e^{iku} \hat{J}''(b(u)) \hat{R}^u(1), \quad (117)$$

from Eq. (77) and the fact that  $q_{k \neq 0} = 0$  it follows that  $\hat{R}_{k \neq 0}(\alpha) = 0$ , and hence also  $D_{k \neq 0} = 0$ . Thus

$$D_k = \frac{2}{T} \hat{J}''(b) \hat{R}_0(1) \delta_{k,0}. \quad (118)$$

Furthermore, Eq. (76) with  $\alpha=1$  and  $k=0$  gives

$$(\mu + \Sigma) \hat{R}_0(1) = -2D_0(\tilde{q}_0 - q_0) = -\frac{4}{\mu + \Sigma} \hat{J}''(b) \hat{R}_0(1), \quad (119)$$

which can be written as

$$0 = \hat{R}_0(1) \left[ 1 + \frac{4}{(\mu + \Sigma)^2} \hat{J}''(b) \right]. \quad (120)$$

Equation (120) with  $\hat{R}_0(1) \neq 0$  implies the marginal stability condition

$$-\left(\frac{T}{L}\right)^2 = \left(\frac{2T}{L(\mu + \Sigma)}\right)^2 \hat{J}''\left(a(0) + \frac{2T}{L(\mu + \Sigma)}\right). \quad (121)$$

Equations (115), (144), (145), and (121) fully solve the model. These equations, with the exception of Eq. (121), are the same as the equations for the statics that we obtained using the replica method in Refs. [6,12]. Equation (121), though, is of pure dynamic origin and differs from the corresponding equation obtained in the replica method by extremizing the variational free energy with respect to the RSB break point  $x_c$ . We give here for comparison the corresponding equation obtained in the statics replica calculation [12]:

$$\frac{L^2 x_c^2}{T^2} \left\{ \hat{J}\left(a(0) + \frac{2T}{L(\mu + \Sigma)}\right) - \hat{J}\left(a(0) + \frac{2T}{L(\mu + \Sigma)}\left(1 + \frac{\Sigma}{\mu x_c}\right)\right) \right\} + \frac{2Lx_c}{T} \frac{\Sigma}{\mu(\mu + \Sigma)} \hat{J}'\left(a(0) + \frac{2T}{L(\mu + \Sigma)}\right) \times \left(1 + \frac{\Sigma}{\mu x_c}\right) - \frac{\Sigma}{\mu + \Sigma} + \ln\left(1 + \frac{\Sigma}{\mu}\right) = 0. \quad (122)$$

Equation (121) can be obtained in the replica calculation by requiring marginal stability—i.e., the condition of a vanish-

ing replicon mass—but it does not correspond to the optimized variational solution.

## XI. SOLVING THE EQUATIONS

In this section we discuss the solution to the equations derived in the last section. Although a full solution can be found numerically, our goal here is to go as far as one can with analytic methods because they give a better understanding of the nature of the solution. An analytical solution becomes possible for a system of large volume (small value of  $\mu$ ) and when the polymer is very long (large  $L$ ).

Before we consider the solution of the equations derived above for a polymer, let us first discuss the special limit where the problem reduces to a classical particle in a random potential. Looking back at the original Hamiltonian one realizes that in the limit  $M \rightarrow \infty$  and  $T \rightarrow TL$  (or alternatively  $L=1$ ) the problem should become the same as the particle problem discussed in Refs. [18,38,39]. Looking back at our equations we see that in the limit  $M \rightarrow \infty$  only the  $k=0$  component of  $\tilde{q}_k$  survives.

In the ergodic case  $E_0$  is given by

$$E_0 = \frac{2}{T} \hat{J}' \left( \frac{2T}{\mu} \right), \quad (123)$$

and the expressions (99) for  $\tilde{q}$  and  $q$  become

$$\tilde{q} = \frac{T}{\mu} + \frac{TE_0}{\mu^2}, \quad (124)$$

$$q = \frac{TE_0}{\mu^2}. \quad (125)$$

These are exactly the same as Eqs. (57)–(59) derived by Engel [38] using the replica method and correspond to his replica-symmetric solution.

In the spin-glass case we again see that as  $M \rightarrow \infty$  we have  $a(u)=0$ . Thus Eqs. (113)–(115) with  $a(0)=0$  and  $L=1$  agree exactly with Eqs. (74)–(76) of Engel [38] using replicas with one-step RSB. However, it is Eq. (122) with  $a(0)=0$  and  $L=1$  that agrees with Engel's Eq. (78) and not the dynamical equation (121) with  $a(0)=0$ , which agrees with the dynamical equation derived in Refs. [18,39]. We also find that

$$\tilde{q} = \frac{T}{\mu + \Sigma} \left( 1 + \frac{E_0}{\mu} + \frac{E_{0,0}\Sigma}{\mu^2} \right) \quad (126)$$

agrees with Engel's Eq. (79).

Let us review Engel's solution to the variational equations for the short-range, random potential case in the limit of small  $\mu$  (but still  $\mu \neq 0$ ). Equation (122) with  $a(0)=0$  and  $L=1$  gives

$$\frac{x_c^2}{T^2} \hat{J} \left( \frac{2T}{\Sigma} \right) - \ln(\mu) = 0. \quad (127)$$

Thus

$$x_c = T \left( \frac{|\ln(\mu)|}{|\hat{J}(0)|} \right)^{1/2}, \quad (128)$$

where we used the fact that  $2T/\Sigma \rightarrow 0$  for  $\mu \rightarrow 0$  as will become clear shortly. Notice that in this case of a particle in a random potential we must consider a random potential that is regularized at small distances (not a strict  $\delta$  function), so  $\hat{J}(0)$  is properly defined. Equation (115) then gives

$$\Sigma = \frac{2x_c}{T} \hat{J}'(0) = 2\hat{J}'(0) \left( \frac{|\ln(\mu)|}{|\hat{J}(0)|} \right)^{1/2}. \quad (129)$$

Also  $E_{0,0}$  vanishes like

$$E_{0,0} = \frac{2}{T} \hat{J}' \left( \frac{2T}{\mu x_c} \right) \sim \frac{g}{T(2\pi)^{d/2}} \left( \frac{\mu^2 |\ln(\mu)|}{2|\hat{J}(0)|} \right)^{(2+d)/4} \quad (130)$$

and  $E_0 = 2\hat{J}'(0)/T$ . Thus

$$\tilde{q} \approx \frac{TE_0}{\Sigma\mu} \sim \frac{|\hat{J}(0)|^{1/2}}{\mu\sqrt{|\ln(\mu)|}}. \quad (131)$$

The importance of Eq. (131) is that it is consistent with the following Imry-Ma-type argument: Since the random potential has a Gaussian distribution of variance  $g$ , in a volume of radius  $r_g$  the minimum of the potential is given by  $V_m \sim -\sqrt{g \ln(r_g)}$ . This result follows from the fact that when we pick  $r_g^d$  numbers from a Gaussian distribution with zero mean and unit variance, the lowest expected value of the potential is given by the solution of the equation

$$\int_{-\infty}^{V_m} \exp(-V^2/2g) \sim \frac{1}{r_g^d}. \quad (132)$$

Thus the Hamiltonian is  $H(x_g) \sim (1/2)\mu r_g^2 - \sqrt{g \ln r_g}$  and minimization gives

$$r_g \sim \frac{g^{1/4}}{\sqrt{\mu} |\ln \mu|^{1/4}}, \quad (133)$$

yielding

$$\tilde{q} = \langle x^2 \rangle \sim \frac{g^{1/2}}{\mu\sqrt{|\ln \mu|}}. \quad (134)$$

This is a nonanalytic expression that cannot be obtained from perturbation theory. The critical temperature below which the RSB solution is the stable one is given by the condition  $x_c = 1$  or  $T_c = [\hat{J}(0)/|\ln \mu|]^{1/2}$ .

In the dynamics approach though, Eq. (121) replaces Eq. (122). For small  $\mu$  since  $\Sigma \gg \mu$ , Eq. (121) gives

$$-1 = \frac{4}{\Sigma^2} \hat{J}''(0) \quad (135)$$

or

$$\Sigma = 2|\hat{J}''(0)|^{1/2}. \quad (136)$$

Again  $E_0 = 2\hat{J}'(0)/T$  and  $E_{0,0}$  vanishes like  $\mu^{(2+d)/2}$ , and thus from Eq. (115) we get

$$x_c = T \frac{|\hat{J}''(0)|^{1/2}}{\hat{J}'(0)}. \quad (137)$$

We also find that

$$\tilde{q} \approx \frac{TE_0}{\Sigma\mu} \sim \frac{\hat{J}'(0)}{\mu|\hat{J}''(0)|^{1/2}}. \quad (138)$$

We observe that in the dynamical formulation  $\ln \mu$  is replaced by the constant  $\hat{J}''(0)|\hat{J}(0)|/(\hat{J}'(0))^2$  which is equal to  $(d+2)/d$  for the short-ranged correlated potential. The Imry-Ma result is not satisfied, even though the distance of the particle from the origin still diverges for  $\mu \rightarrow 0$ . This is probably due to the fact that in the dynamical formulation the particle is impeded by large barriers to search the entire volume as effectively for the lowest minimum of the random potential as is obtained in the statics, even in the limit of large times. The fact that the dynamical solution differs from the statics in this case was observed in Ref. [18], but they did not compare the solutions in detail. They observe that in the dynamical solution the free energy is higher than the replica free energy obtained in the statics. This discrepancy may also be due to the fact that the initial conditions of the dynamics are completely random and are not weighted with an appropriate Boltzmann factor [40].

We now return to the polymer problem and discuss the solution to the previously derived equations for the case of a random potential with short-range correlations characterized by  $\hat{\gamma} = 1 + d/2$ . The ergodic phase for large times in principle should correspond to the replica-symmetric solution of the statics. In the equilibrium solution such a solution arises in two situations: First, in the infinite-volume limit, when  $\mu = 0$ . In this case the polymer totally collapses to a size of a single monomer, since deep minima of unbounded bottom exist in this limit. This case was discussed before in Ref. [12] and it was shown that the solution corresponds to the case of an annealed disorder. Since the solution was discussed before, we will not repeat it here. In this case the equations for the dynamics are the same as obtained in the replica solution. Second, for the finite-volume case the ergodic solution is applicable when  $L < L_c$  where  $L_c$  is the value for which the spin-glass solution ceases to exist; see below. It turns out that  $L_c$  is a number of order unity; thus, except for very short chains, the nonergodic solution is the correct one. It is a subtle question if the domain of validity of the ergodic phase in the dynamics and in the statics coincide. For the case of infinite volume one has to take the limit  $\mu \rightarrow 0$  before the limit of large time. We will not discuss this question further here.

In the Appendix we give the solution of the ergodic equations for a special case of a long-range random potential with quadratic correlations and show that it gives the same solution as previously obtained by Shiferaw and Goldschmidt [13] using a different method. Of course for a quadratically

correlated potential the variational approximation is exact.

We proceed to discuss the solution to the equations in the spin-glass case when the random potential has short range correlations ( $\hat{\gamma} = 1 + d/2$ ). As mentioned above, when the volume is large but finite the spin-glass solution is the appropriate solution when  $L$  is large enough [12]. For the spin-glass case we have seen that one of the equations of the dynamics differs from one of the equations derived in the statics using the replica method. We now rederive the solution of the equations of the statics and then show how the solution of the equations of the dynamics differs from it. The reason that we reconsider the solution of the equations of the statics is first because the variational scheme that emerged above is more general than the scheme employed in Ref. [12] since it involves more variational parameters like the scheme used in Ref. [6], and we want to show that the end results are the same. Second, the steps of the solution will facilitate the solutions of the dynamical equations.

In the limit of small  $\mu$  (but still  $\mu \neq 0$ ), Eq. (122) gives

$$\frac{x_c^2 L^2}{T^2} \hat{J} \left( a(0) + \frac{2T}{L\Sigma} \right) - \ln(\mu) = 0. \quad (139)$$

Thus

$$x_c = \frac{T}{L} \left( \frac{|\ln(\mu)|}{|\hat{J}(a(0))|} \right)^{1/2}, \quad (140)$$

where we used the fact that  $2T/(L\Sigma) \ll a(0)$  for large  $L$  (and even for finite  $L$  as  $\mu \rightarrow 0$  as can be checked *a posteriori* with full solution). Equation (115) then gives

$$\Sigma = \frac{2Lx_c}{T} \hat{J}'(a(0)) = 2\hat{J}'(a(0)) \left( \frac{|\ln(\mu)|}{|\hat{J}(a(0))|} \right)^{1/2}, \quad (141)$$

and from Eq. (114) one obtains

$$E_{0,0} = \frac{2Lx_c}{T} \hat{J}' \left( \frac{2T}{\mu L x_c} \right) \sim \frac{gL}{T(2\pi)^{d/2}} \left( \frac{\mu^2 |\ln(\mu)|}{4|\hat{J}(a(0))|} \right)^{(2+d)/4}. \quad (142)$$

It is convenient to define the variables

$$\lambda_k = F_k - E_0 + \Sigma + \mu, \quad k \neq 0. \quad (143)$$

Using Eq. (113),  $\lambda_k$  satisfy the equation

$$\lambda_k = \mu + \Sigma + \frac{2}{T} \int_0^L du \left\{ (1 - e^{iku}) \hat{J}'(a(0) - a(u)) - \hat{J}' \left( a(0) + \frac{2T}{L(\mu + \Sigma)} \right) \right\}. \quad (144)$$

and in terms of  $\lambda_k$  the constant  $a(u)$  is given by

$$a(u) = \frac{2T}{L} \sum_{k \neq 0} \frac{e^{-iku}}{Mk^2 + \lambda_k} \rightarrow 2T \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{-iku}}{Mk^2 + \lambda_k}, \quad (145)$$

where the last expression is valid for large  $L$ . The integral on the right-hand side (RHS) of Eq. (144) converges as  $L$  becomes large, and thus to leading order the parameters  $\lambda_k$  are

$O(1)$  with respect to  $L$ . To simplify the integral further notice first that the integrand is invariant under the transformation  $u \rightarrow L-u$ , and thus it symmetric about  $u=L/2$ . The total integral is twice its value up to  $u=L/2$ , and  $\lambda_k$  can consistently be taken as real. For large  $L$  we obtain

$$\lambda_k = \mu + \Sigma - \frac{4}{\Sigma} \hat{J}''(a(0)) + \frac{4}{T} \int_0^\infty du (1 - \cos ku) \times \{\hat{J}'(a(0) - a(u)) - \hat{J}'(a(0))\} + O(1/L), \quad (146)$$

where  $\Sigma$  is given by Eq. (141) and

$$a(u) = 2T \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{\cos ku}{Mk^2 + \lambda_k}. \quad (147)$$

Note that we have added to the integral in Eq. (146) a term

$$\frac{4}{T} \hat{J}'(a(0)) \int_0^L du \cos(ku) = 0, \quad k \neq 0. \quad (148)$$

Our goal is to characterize the behavior of the end-to-end distance of the polymer that can be extracted from the correlation function

$$\begin{aligned} \tilde{b}(L) &= \lim_{t \rightarrow \infty} \langle [x(L,t) - x(0,t)]^2 \rangle = \frac{2}{L} \sum_{k \neq 0} (1 - e^{-ikL}) \tilde{q}_k \\ &= a(0) - a(L). \end{aligned} \quad (149)$$

We are now going to argue that as  $\mu \rightarrow 0$  (but still finite),  $\lambda_k$  satisfies, to leading order in  $|\ln \mu|$ , the scaling form

$$\lambda_k = (g|\ln \mu|)^{\delta} \tilde{\lambda} + \frac{(g|\ln \mu|)^{\delta}}{|\ln \mu|} f(k(g|\ln \mu|)^{-\delta/2}), \quad k > 0, \quad (150)$$

where  $\delta=4/(4-d)$  and  $\lambda_{-k}=\lambda_k$ . For small  $k$  the function  $f$  satisfies  $f(x) \sim x^2$  and is regular around 0. For large  $x$ , it can be shown that  $f(x) \sim x^{d/2}$ . Substituting Eq. (150) into Eq. (147) and changing the integration variable

$$k \rightarrow k(g|\ln \mu|)^{\delta/2}, \quad (151)$$

we find

$$a(u) = \frac{2T}{(g|\ln \mu|)^{\delta/2}} \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{\cos[k(g|\ln \mu|)^{\delta/2}u]}{Mk^2 + \tilde{\lambda} + \frac{1}{|\ln \mu|} f(k)}, \quad (152)$$

from which it follows that, to leading order in  $|\ln \mu|$ ,

$$a(u) = \frac{T}{\sqrt{\tilde{\lambda}M}} (g|\ln \mu|)^{-\delta/2} e^{-\sqrt{\tilde{\lambda}M}(g|\ln \mu|)^{\delta/2}u} \quad (153)$$

and in particular  $a(0) = T/\sqrt{M\tilde{\lambda}}(g|\ln \mu|)^{-\delta/2}$ . Substituting on the RHS of Eq. (146) and changing the integration variable

$$u \rightarrow u(g|\ln \mu|)^{-\delta/2}(M/\tilde{\lambda})^{1/2}, \quad (154)$$

we find, for small  $\mu$ ,

$$\begin{aligned} \lambda_k &= \frac{d^{1/2}}{(2\pi)^{d/4}} \left( \frac{T}{\sqrt{M\tilde{\lambda}}} \right)^{-(d+4)/4} (g|\ln \mu|)^{1/2 + \delta(d+4)/8} \\ &\times \left[ 1 + O\left( \frac{1}{|\ln \mu|} \right) \right] + \frac{(g|\ln \mu|)^{1+\delta/4}}{|\ln \mu|} f(k(g|\ln \mu|)^{-\delta/2}), \end{aligned} \quad (155)$$

where

$$\begin{aligned} f(x) &= \frac{2M}{(2\pi)^{d/2}T^2} \left( \frac{T}{\sqrt{M\tilde{\lambda}}} \right)^{-d/2} \int_0^\infty du \{1 - \cos[x(M/\tilde{\lambda})^{1/2}u]\} \\ &\times \left\{ \frac{1}{(1 - e^{-u})^{d/2+1}} - 1 \right\}. \end{aligned} \quad (156)$$

We see that for consistency  $\delta$  must satisfy  $\delta=4/(4-d)$ . We also see that the parameter  $\tilde{\lambda}$  is given by

$$\tilde{\lambda} = \left( \frac{d}{(2\pi)^{d/2}} \right)^{4/(4-d)} \left( \frac{M}{T^2} \right)^{(4+d)/(4-d)}. \quad (157)$$

Using Eq. (153) we have

$$\tilde{b}(L) = \frac{T}{\sqrt{\tilde{\lambda}M}} (g|\ln \mu|)^{-\delta/2} (1 - e^{-\sqrt{\tilde{\lambda}M}(g|\ln \mu|)^{\delta/2}L}). \quad (158)$$

Thus the end-to-end distance is given by

$$R^2 \sim \frac{T}{\sqrt{\tilde{\lambda}M}} (g|\ln \mu|)^{-\delta/2}, \quad (159)$$

from which it follows that

$$R \sim (g|\ln \mu|)^{-1/(4-d)} \sim (g|\ln \nu)^{-1/(4-d)}. \quad (160)$$

Here  $\nu$  is the total volume and this result has the correct  $g$  dependence and the subtle  $\ln \nu$  dependence as was originally argued by Cates and Ball [11] and derived in Ref. [12] using the replica method.

The important observation is that when  $k$  increases from 0 to  $\sqrt{\tilde{\lambda}M}(g|\ln \mu|)^{\delta/2}$ ,  $\lambda_k$  changes only by a factor of  $1 + O(1/|\ln \mu|)$  and thus the results of the single  $\lambda$  parameter used in the variational scheme of Ref. [12] remain intact. The effective mass of the low-lying nonzero modes is approximately given by  $\sqrt{\lambda_k} \sim \sqrt{\Sigma}$ .

The above scaling arguments are valid provided  $a(0) > d\sigma^2/2$  [see Eq. (34)], which is what we are going to assume. Of course for a  $\delta$ -function correlated random potential  $\sigma \rightarrow 0$  and this condition always holds. For this condition to apply in the case of other short-range correlated random potentials,  $g$ , the variance of the disorder, may not be too large, such that the size of the polymer is not smaller than the correlation length of the disorder.

The parameters  $\Sigma$  and  $x_c$  are given to leading order in  $L$  and  $|\ln \mu|$  by

$$\Sigma = d^{4/(4-d)} (2\pi)^{-2d/(4-d)} (M/T^2)^{(4+d)/(4-d)} (g|\ln \mu|)^{4/(4-d)}, \quad (161)$$

$$x_c = \frac{1}{L} \left( \frac{d^{d-2}}{(2\pi)^d} g^2 T^{-(d+4)} M^d |\ln \mu|^{d-2} \right)^{-1/(4-d)}. \quad (162)$$

We see that  $x_c < 1$  for large enough  $L$  and for  $d > 2$  for fixed  $L$  and small enough  $\mu$ . The value of  $L$  corresponding to  $x_c = 1$  is denoted by  $L_c$ . The condition  $x_c < 1$  can be written as

$$\ln \mathcal{V} < L \sqrt{\frac{g \ln \mathcal{V}}{R^d}}, \quad (163)$$

which means that the translational entropy is smaller than the binding energy in the typical minimum of the random potential, thus implying that the polymer is truly localized. It also follows that  $x_c < 1$  is equivalent to the condition

$$\frac{gL^{(4-d)/2}}{T^2 b_K^d} (\ln \mathcal{V})^{(d-2)/2} > 1, \quad (164)$$

up to some unimportant numerical factor.

We now discuss the solution of the dynamical equations. Equation (121) replaces Eq. (122). For small  $\mu$  since  $\Sigma \gg \mu$  Eq. (121) gives

$$-1 = \frac{4}{\Sigma^2} \hat{J}''(a(0)) \quad (165)$$

or

$$\Sigma = 2|\hat{J}''(a(0))|^{1/2}. \quad (166)$$

From Eq. (115) we get

$$x_c = \frac{T|\hat{J}''(a(0))|^{1/2}}{L\hat{J}'(a(0))}, \quad (167)$$

and we see that there is no longer any  $\ln \mu$  dependence as in the replica solution. Again  $x_c < 1$  for large enough  $L$ . Using this equation we see that

$$E_{0,0} = \frac{2L}{T} \hat{J}' \left( \frac{2T}{\mu L x_c} \right) \sim \frac{gL}{T(2\pi)^{d/2}} \left( \frac{\mu^2 |\hat{J}''(a(0))|}{4\hat{J}'(a(0))^2} \right)^{(2+d)/4}. \quad (168)$$

The equation for  $\lambda_k$  again reads

$$\lambda_k = \mu + \Sigma - \frac{4}{\Sigma} \hat{J}''(a(0)) + \frac{4}{T} \int_0^\infty du (1 - \cos ku) \times \{\hat{J}'(a(0) - a(u)) - \hat{J}'(a(0))\} + O(1/L). \quad (169)$$

Since in this case there is no  $\ln \mu$  dependence, we can repeat the scaling of  $\lambda_k$  with  $g$  as before but not with  $\ln \mu$ . We find the  $g$  dependence as before; thus,

$$\lambda_k = g^{4/(4-d)} \tilde{\lambda}_k, \quad (170)$$

where  $\tilde{\lambda}_k$  are independent of  $g$ . We also find that since  $|\ln \mu|$  is absent, the parameter  $M$  will be shifted by contributions from  $f(k)$ . We can still argue that  $\lambda_k$  are independent of  $L$ , which is an important result, and that they have the  $g$  dependence described above. This implies that

$$R \sim g^{-1/(4-d)} \quad (171)$$

independent of  $L$ . Similarly we find that  $\Sigma$  is finite and satisfies  $\Sigma \sim g^{4/(4-d)}$  and  $x_c \sim (1/L)g^{-2/(4-d)}$ . This last condition implies that the polymer is localized whenever  $x_c < 1$  which means approximately  $gL^{(4-d)/2} > 1$ .

These results still contain important physics—namely, that the size of the polymer is independent of its length and also it has the correct dependence on the strength of the disorder, but the more subtle  $\ln \mathcal{V}$  dependence resulting from a sophisticated Imry-Ma-type argument is missing.

## XII. DISCUSSION OF THE DYNAMICAL BEHAVIOR

Here we make some further comments about the dynamical behavior of the polymer in the spin-glass phase. Consider the motion of the center of mass of the polymer:

$$\begin{aligned} B_{c.m.}(t, t') &\equiv \langle \langle [x_{c.m.}(t) - x_{c.m.}(t')]^2 \rangle \rangle / d \\ &= \frac{1}{dL^2} \int_0^L du \int_0^L du' \langle \langle [x(u, t) - x(u, t')] \\ &\quad \times [x(u', t) - x(u', t')] \rangle \rangle \\ &= \frac{1}{L} [C_0(t, t) + C_0(t', t') - 2C_0(t, t')]. \end{aligned} \quad (172)$$

Let us denote  $t' = t_W$  and  $\tau = t - t'$ . In the TTI regime for large  $t_W$  and  $\tau \ll t_W$ , as  $\tau$  increases,  $B_{c.m.}(\tau)$  increases from 0 to

$$b^{(1)} = \lim_{\tau \rightarrow \infty} \lim_{t_W \rightarrow \infty} B_{c.m.}(t_W + \tau, t_W) = \frac{2}{L} (\tilde{q}_0 - q_0) = \frac{2T}{L(\mu + \Sigma)}. \quad (173)$$

In this regime the FDT also holds. Let us compare this behavior with the behavior of a free chain. For a free chain, for large  $t, t'$  we have

$$B_{c.m.}(\tau) = \frac{2T}{L\mu} (1 - e^{-\mu\tau}) \xrightarrow{\tau \rightarrow \infty} \frac{2T}{L\mu}, \quad (174)$$

and thus when disorder is present  $\Sigma$  is replacing  $\mu$  when  $\mu \rightarrow 0$  and we observe that  $b^{(1)}$  remains finite as  $\mu \rightarrow 0$ . In this regime the polymer becomes trapped inside a local potential minimum and we see that for large  $\tau$  (but still less than  $t_W$ ) there is no diffusion, even though for very small  $\tau$  we expect diffusive behavior. Thus for large enough  $\tau$  there will be a plateau in the plot of  $B_{c.m.}(\tau)$  as a function of  $\tau$ . In this regime the short-time estimates of previous investigations (see, e.g., Ref. [25]) should be valid.

However, for large and fixed  $t_W$  as  $\tau$  becomes sufficiently large,  $B_{c.m.}$  will leave the plateau and continue to grow above  $b^{(1)}$  until it reaches the value

$$\begin{aligned} b^{(0)} &= \lim_{\tau \rightarrow \infty} B_{c.m.}(t_W + \tau, t_W) = \frac{2}{L} (\tilde{q}_0 - q_{0,0}) \\ &= \frac{2T}{L(\mu + \Sigma)} \left( 1 + \frac{\Sigma}{\mu x_c} \right). \end{aligned} \quad (175)$$



The size of the plateau depends on  $t_w$ . The larger  $t_w$ , the larger the plateau and the larger value of  $\tau$  required for  $B_{c.m.}$  to increase beyond  $b^{(1)}$ . Thus the polymer does not remain trapped forever but eventually hops to another minimum of the potential. Notice also that as  $\mu \rightarrow 0$ ,  $b^{(0)} \rightarrow 2T/(\mu L x_c) \sim 2Tg^{2/(4-d)}/\mu$ . This value is independent of  $L$  and should represent the typical square of the hopping distance of the polymer among different local minima of the potential. The larger the waiting time, the deeper the local minimum occupied by the polymer and hence the longer it takes it to hop to another minimum. From with this observation we still lack an estimate of the time dependence of  $B_{c.m.}(t, t')$  in the vicinity of  $b^{(1)}$  and  $b^{(0)}$ . A more detailed calculation is needed to derive the asymptotic growth rate in the different time regimes, and this will be left for future work.

Another quantity of interest is the single segment (bead or monomer) mean square displacement

$$\begin{aligned} B^{u=0}(t, t') &= \langle \langle [x(s, t) - x(s, t')]^2 \rangle \rangle / d \\ &= \frac{1}{L} \sum_k [C_k(t, t) + C_k(t', t') - 2C_k(t, t')]. \end{aligned} \quad (176)$$

In this case the asymptotic mean-square displacement in the TTI regime becomes

$$b_{seg}^{(1)} = 2(\tilde{q} - q) = b^{(1)} + a(0) \quad (177)$$

and the asymptotic value in the aging regime becomes

$$b_{seg}^{(0)} = 2(\tilde{q}_0 - q_0) = b^{(0)} + a(0), \quad (178)$$

with  $b^{(1)}$  and  $b^{(0)}$  defined above and

$$a(0) \sim (Tb_K^2)^{(4/(4-d))} g^{-2/(4-d)}. \quad (179)$$

The quantity  $b_{seg}^{(1)}$  is dominated by  $a(0)$  and thus the size of the mean-square displacement of a bead is the same as the square of the end-to-end distance. The quantity  $b_{seg}^{(0)}$  remains essentially the same as  $b^{(0)}$  for small  $\mu$ .

The results of this section—i.e., the width of a localized state of the polymer and the average distance squared between different localized states—is the same as discussed in the interpretation of the one-step RSB solution in Ref. [13], Sec. VII, with only the subtle dependence on  $\ln \mathcal{V}$  missing.

### XIII. CONCLUSIONS

In this paper we derived the dynamical equation for a Gaussian chain in a short-range correlated random potential. We used a simple Langevin dynamics, and we discovered that there are two possible scenarios at large times: a stationary regime where the FDT applies and an aging regime where the FDT breaks down at large time separation. In the aging regime the FDT can be shown to be replaced by a modified or generalized form commonly referred to as the GFDT and involves a parameter  $x_c$  similar to Parisi's parameter for one-step RSB. Only one-step RSB is necessary for the case of short-range correlations. (In the long-range case that was discussed in Ref. [6] in the equivalent context of a

quantum particle in a random potential full RSB applies.)

The stationary regime represents the dynamics of a chain trapped in a local minima of the random potential. Eventually after very long time the chain can escape from its pinning and hop to another minimum elsewhere, leading to a history dependence and violation of equilibrium theorems. In the long-time limit, for a short-ranged correlated random potential, the dynamical equations become identical to the equations of the statics as derived from the replica method except for one equation that involves  $x_c$ , which is different from the equation derived in the statics using replicas. This is probably due to the fact that starting from random initial conditions the dynamics gets influenced by large barriers and does not explore the potential landscape as efficiently as to reproduce the statics even at large times. Thus the subtle  $\ln \mathcal{V}$  dependence of the statics that emerges from an Imry-Ma-type argument even for the case of a zero-dimensional object in a random potential is not reproduced by the dynamical equations. Our results are based, of course, on the Gaussian variational approximation but the emerging picture is probably valid.

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### APPENDIX: RANDOM POTENTIAL WITH QUADRATIC CORRELATIONS

In this appendix we discuss the exactly solvable case of a potential with long-range quadratic correlations [14]. Since the variational approximation becomes exact for such a potential, our dynamical equations should reproduce the solution found using the statics and replica formalism. For this case  $J(a) = (1/2)ga + \text{const}$ , and so is  $\hat{J}(a)$  since  $\gamma = 0$ . Thus  $\hat{J}'(z) = g/2$ , a constant. (The notation here is slightly different from Ref. [14] where we used  $J(a) = 2\sigma a + \text{const}$ , so  $g \rightarrow 4\sigma$ .) Only the ergodic case applies in this case since  $E_0 = E_{0,0}$ . We find

$$E_0 = \frac{gL}{T}, \quad (A1)$$

$$F_k = \frac{gL}{T} (1 - \delta_{k,0}). \quad (A2)$$

We also obtain

$$\tilde{q}_k = \frac{gL}{\mu^2} \delta_{k,0} + \frac{T}{\mu + Mk^2}, \quad (A3)$$

$$q_k = \frac{gL}{\mu^2} \delta_{k,0}. \quad (A4)$$

From these results it follows that

$$\tilde{q} = \frac{g}{\mu^2} + \frac{T}{L} \sum_k \frac{1}{Mk^2 + \mu} \rightarrow \frac{g}{\mu^2} + \frac{T}{2\sqrt{M\mu}}, \quad (\text{A5})$$

where the last expression applies for large  $L$  and also

$$q = \frac{g}{\mu^2}. \quad (\text{A6})$$

The correlation function  $\tilde{b}(L)$  is given by

$$\tilde{b}(L) = \frac{1 - e^{-L\sqrt{\mu/M}}}{\sqrt{M\mu}}. \quad (\text{A7})$$

For small  $\mu$  this function becomes equal to  $L/M$  as in the free case. These results coincide with the results obtained in Ref. [14].

Actually in this case one can write a closed form solution for Eqs. (56) and (57). The solution is

$$C_k(t, t') = \frac{gL}{\mu^2} \delta_{k,0} (1 - e^{-\mu t} - e^{-\mu t'} + e^{-\mu(t+t')}) + \left( C_k(0, 0) - \frac{T}{\mu + Mk^2} \right) e^{-(\mu + Mk^2)(t+t')} + \frac{T}{\mu + Mk^2} e^{-(\mu + Mk^2)|t-t'|}, \quad (\text{A8})$$

$$R_k(t, t') = \theta(t - t') e^{-(\mu + Mk^2)(t-t')}. \quad (\text{A9})$$

Indeed for large times  $C(t-t')$  becomes TTI and depends only on the difference  $t-t'$ , and the FDT holds.

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